CONTROLABILITY OF PARABOLIC AND HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS, AND OF THEIR FINITE DIFFERENCE APPROXIMATIONS, WITH RESPECT TO THE SHAPE OF THE DOMAIN

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Abstract. In this article we study a controllability problem for parabolic and hyperbolic partial differential equations in which the control is the shape of the domain where the equation holds. Given a right-hand source term, the quantity to control is the trace of the solution into an open subdomain and at a given time. The mapping that associates this trace to the shape of the domain is non-linear. In this paper we first consider the continuous control problem and show an approximate controllability property for the linearized parabolic problem and an exact local controlability for the hyperbolic problem. Next we address the same questions in the context of a finite difference spatial semi-discretization. We prove a local controllability result for the parabolic problem, and an exact controllability for the hyperbolic one, applying the local surjectivity theorem together with a unique continuation property of the underlying adjoint discrete system.

Key words. partial differential equations, parabolic, hyperbolic, Heat equation, Wave equation, controllability, shape of the domain

AMS subject classifications. 35K05, 93B03, 65M06

Introduction. In this paper we consider parabolic and hyperbolic partial differential equation on an open set $\Omega \subset \mathbb{R}^n$, given a source term. This domain Ω fully contains a simply connected open set ω. We assume that the solution of the partial differential equation restricted to ω at time T is accessible by means of measurement. The problem we address is a control and accessibility problem: *Which solution can we reach at time T when the open set Ω varies?*

This paper follows the work of Chenais and Zuazua [7] about the controllability of an elliptic problem with respect to the shape of the domain. In our paper, the domain can vary in time and the equations are time-dependent, of parabolic and hyperbolic type.

For the sake of clarity, we only study here the heat and wave equation with Dirichlet boundary conditions. Nevertheless, the method we use in this paper is valid for different parabolic or hyperbolic systems and directly applies for an elliptic equations. The same framework can also be used to study other boundary conditions, but the complexity of the problem increases. For instance for the same problem with Neumann boundary conditions require in the semi-discrete case a carefull study which is out of the scope of the present paper.

The controllability problem is solved in the first section, and we show an approximate controllability property for the linearized parabolic problem and an exact local controlability for the hyperbolic problem.

In the second section we address the same questions for the finite-difference space semi-discretized problem. Performing a finite-difference spatial discretization on the heat and wave equations, we address the same question as in the continuous-space case: *is the solution of the semi-discrete equation controllable with respect to the shape of the (time-dependent) discrete domain?*. The answear is yes, since we prove that

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both the parabolic and the hyperbolic semi-discrete problems are exactly controllable.

1. Control of the continuous wave and heat equations. In this section, we solve the continuous-time problem, and show that the parabolic linearized problem is approximately controllable, while the hyperbolic problem is exactly controllable.

1.1. Position of the problem. Let $\mathcal{L}$ be a parabolic or hyperbolic differential operator, $f \in L^2(\mathbb{R}^n)$, $T > 0$ the time when we want to control the equation, and Ω_0 an open regular domain of $\mathbb{R}^n$.

Let y_0 be the solution (when it is uniquely defined) of:

$$\begin{cases} \mathcal{L}y_0(x,t) = f(x) & \forall (x,t) \in \Omega_0 \times \mathbb{R}^+ \\ y_0(x, t=0) = 0 & \forall x \in \mathbb{R}^n \\ y_0(x,t) = 0 & \forall (x,t) \text{ in } \partial\Omega_0 \times \mathbb{R}^+. \end{cases} \tag{1.1}$$

Let now $\mathcal{V}\big(y_0(\cdot,T)\big)$ be a neighborhood of the function $y_0|_\omega(\cdot,T)$ in $H^1(\omega)$. In this paper we are interested in finding small dynamical perturbations of the open set Ω_0 so that we can reach at time T any function $y_d \in H^1_0(\omega) \cap \mathcal{V}(y_0(T))$.

This problem is a control and accessibility problem, where the control is the shape of the domain. Many problems of this type are solved by optimization methods (see for instance [4, 6, 5, 14, 16, 17, 19]). In this case the problem is written in the form:

$$\inf_{\Omega \in \mathcal{U}} \|y_\Omega - y_d\|$$

where $\mathcal{U}$ is the set of domains we take into account. Under suitable conditions on the set of admissible domains $\mathcal{U}$, the existence of the minimizer can be guaranteed. Note however that this existence result in itself does not provide any information on whether $y_\Omega|_\omega = y_d$ does actually hold or not for a suitable choice of the domain Ω, or even about the minimal distance between $y_\Omega|_\omega$ and y_d. Therefore, optimization techniques will not solve the controllability problem under consideration.

There are several versions of the controllability property that make sense in the situation under consideration and that we shall address. Indeed, the problem may be formulated at least in two different ways, as an exact or approximate controllability problem:

(i). The *exact* controllability: it is the case when one can find an open set $\Omega^*(t) \in \mathcal{U}$ such that $y_{\Omega^*}(T)|_\omega = y_d$.

(ii). The *approximate* controllability: for any fixed $\varepsilon > 0$, is it possible to find $\Omega^*_\varepsilon(t) \in \mathcal{U}$ such that for some norm on a certain set of functions we have

$$\| \, y_{\Omega^*_\varepsilon}(T)\big|_\omega - y_d\| \leq \varepsilon?$$

There is an extensive literature on exact and approximate controllability problems for partial differential equations (PDE) but very little has been done in the context in which the control is the shape of the domain. The interested reader may find an interesting introduction to these topics in J. L. Lions [11] and an updated survey article in E. Zuazua [20, 21].

1.2. Controllability of the heat equation. In this section, we consider the classical heat equation with a source term $f \in L^2(\mathbb{R}^+, \mathbb{R}^n)$ on the open set Ω_0, whose frontier denoted by Γ_0 is $W^{2,\infty}$:

$$\begin{cases} \partial_t y_0(t,x) - \Delta y_0(t,x) & = f(t,x) \quad \forall x \in \Omega_0 \text{ and } \ t>0 \\ y_0(t=0,x) & =0 \quad \forall x \in \Omega_0 \\ y_0|_{\partial\Omega_0}(t) & =0 \quad \forall t>0 \end{cases} \tag{1.2}$$

This equation defines a unique solution y_0, and we will perturb the domain Ω_0 to reach solutions in the neighborhood of $y_0(T)|_\omega$ in $H^1(\omega)$

To this purpose, let us define the set $\mathcal{U}$ of admissible domains we take into account in the control problem.

Definition 1.1. *We denote by $W^{k,\infty}(\mathbb{R}^n,\mathbb{R}^m)$ the set of functions k times differentiable of $\mathbb{R}^n$ (in the sense of distributions), taking its values in $\mathbb{R}^m$, all the differentials being in $L^\infty(\mathbb{R}^n,\mathbb{R}^m)$.*

We work in the standart setting for differentiation with respect to the domain (see for instance [14, 16, 18, 17, 19]). The "admissible" perturbed open sets are the "small dynamical perturbations" of Ω_0. Mathematically speaking, let $\mathcal{V}$ a neighborhood of 0 in the space of continuous and bounded functions $L^\infty(\mathbb{R}^+,W^{2,\infty}(\mathbb{R}^n,\mathbb{R}^n))\bigcap C(\mathbb{R}^+;,W^{2,\infty}(\mathbb{R}^n,\mathbb{R}^n))$ such that for all $\varphi\in\mathcal{V}$ and for all $t\in[0,T]$, the maps $(id+\varphi(t))$ and $(id+\varphi(t))^{-1}$ are homemorphisms of $W^{2,\infty}(\mathbb{R}^n,\mathbb{R}^n)$. This set of φ functions is now denoted by $\mathcal{W}$.

For $\varphi\in\mathcal{V}$, let

$$\Omega_\varphi(t) = (id+\varphi(t))\,(\Omega_0) = \{x+\varphi(t)(x); x\in\Omega_0\}.$$

The perturbed equation is:

$$\begin{cases} \partial_t y_\varphi(t,x) - \Delta y_\varphi(t,x) & = f(t,x) \quad \forall x \in \Omega_\varphi(t) \text{ and } \ t>0 \\ y_\varphi(t=0,x) & =0 \quad \forall x \in \Omega_0 \\ y_\varphi|_{\Gamma_\varphi(t)}(t) & =0 \quad \forall t>0 \end{cases} \tag{1.3}$$

We finally denote by Λ the non-linear map defined by:

$$\Lambda := \begin{cases} \mathcal{W} & \mapsto H^1(\mathbb{R}^n) \\ \varphi & \mapsto y_\varphi(T) \end{cases} \tag{1.4}$$

and $\mathcal{R}(\Lambda)=\{y_\varphi(T)\,|\varphi\in\mathcal{W}\}$ the accessible states at time T of the perturbed equation.

1.2.1. Existence and uniqueness of solutions for the perturbed problem. In this section we use the variational formulation and show weak existence and uniqueness of the solution of the perturbed system.

Theorem 1.2. *The perturbed problem* (1.3) *is equivalent to the following partial differential equation:*

$$\begin{cases} \partial_t\overline{y_\varphi} - \dfrac{1}{|det(id+\nabla\varphi(t))|}div(B(\varphi)\nabla\overline{y_\varphi}) = f\circ(id+\varphi) & on\ \Omega_0\times\mathbb{R}^+ \\ \overline{y_\varphi}(t=0)=0 & on\ \Omega_0 \\ \overline{y_\varphi}|_{\Gamma_0}=0 & \forall t>0 \end{cases} \tag{1.5}$$

where

$$B(\varphi) = \Big|det\big(id+\nabla\varphi(t)\big)\Big| \ \left(\Big[\big(\nabla(id+\varphi(t))\big)^*\Big]^{-1}\right)^* \ \Big[\big(\nabla(id+\varphi(t))\big)^*\Big]^{-1}, \tag{1.6}$$

the star denotes the adjoint and

$$\overline{y_\varphi}(t,x) = y_\varphi\Big(t, x+\varphi(t)(x)\Big).$$

Proof. We use the variational formulation of the equations (1.3) and show that it is equivalent to the variational formulation of the problem (1.5). Let $\phi \in H^1\Big(\mathbb{R}^+, H_0^1(\Omega_\varphi(t))\Big)$. Let Ψ the function defined on $\mathbb{R}^+ \times \Omega_0$ by $\Psi(t,x) = \phi(t, x+\varphi(t)(x))$.
We have:

$$\int_{\mathbb{R}^+\times\Omega_\varphi(t)} (\partial_t y_\varphi \phi - \Delta y_\varphi \phi)\ dt\, dx = \int_{\mathbb{R}^+\times\Omega_\varphi(t)} f\phi\ dt\, dx$$
$$\int_{\mathbb{R}^+\times\Omega_\varphi(t)} (\partial_t y_\varphi \phi + \nabla y_\varphi \nabla \phi)\ dt\, dx = \int_{\mathbb{R}^+\times\Omega_\varphi(t)} f\phi\ dt\, dx$$

Using the change of variable $x = r+\varphi(r)$ with $r \in \Omega_0$. The determinant of the Jacobian matrix of the change of variable is $\det(id+\nabla\varphi)$, so the change of variables theorem gives us the following equivalent expression:

$$\int_{\mathbb{R}^+\times\Omega_0} \{\partial_t \overline{y_\varphi}\Psi - (\nabla y_\varphi)\circ(id+\varphi)\cdot(\nabla\phi)\circ(id+\varphi)\}\,|\det(id+\nabla\varphi(t))|\, dt\, dx$$
$$= \int_{\mathbb{R}^+\times\Omega_0} f\circ(id+\varphi)\Psi\,|\det(id+\nabla\varphi(t))|\, dt\, dx \quad (1.7)$$

Let us now express $(\nabla G)\circ(id+\varphi)$ in function of $\nabla\Big(G\circ(id+\varphi)\Big)$. Let $H = G\circ(id+\varphi)$ and $T = (id+\varphi)$. We have:

$$\begin{aligned}\frac{\partial H}{\partial r_i} &= \frac{\partial((G\circ T)(y)}{\partial r_i}\\ &= \sum_{j=1}^{2}\frac{\partial G}{\partial x_j}\frac{\partial x_j}{\partial r_i}\\ &= (\nabla T)^*\nabla G\end{aligned}$$

where the star denotes the adjoint operator. Indeed, if $x = Tr$,

$$\nabla T = \begin{pmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_1}{\partial r_2} \\ \frac{\partial x_2}{\partial r_1} & \frac{\partial x_2}{\partial r_2}\end{pmatrix},$$

and hence

$$\nabla G = \begin{pmatrix} \frac{\partial G}{\partial x_1} \\ \frac{\partial G}{\partial x_2}\end{pmatrix}.$$

Putting all these results together we get:

$$\int_{\mathbb{R}^+\times\Omega_0} \partial_t \overline{y_\varphi}\Psi - \Big[\big(\nabla(id+\varphi(t))\big)^*\Big]^{-1}\nabla\overline{y_\varphi}\Big[\big(\nabla(id+\varphi(t))\big)^*\Big]^{-1}\nabla\Psi\,|\det(id+\nabla\varphi(t))|\,dt\,dx$$
$$= \int_{\mathbb{R}^+\times\Omega_0} f\circ(id+\varphi)\Psi\,|\det(id+\nabla\varphi(t))|\,dt\,dx \quad (1.8)$$

Finally, we get the following variational problem:

$$\int_{\mathbb{R}^+\times\Omega_0} \left\{\partial_t\overline{y_\varphi}\;|\det(id+\nabla\varphi(t))| - \operatorname{div}\big(B(\varphi)\big)\nabla\overline{y_\varphi}\right\}\Psi dt\,dx$$
$$= \int_{\mathbb{R}^+\times\Omega_0} f\circ(id+\varphi)\Psi\,\Big|\det\big(id+\nabla\varphi(t)\big)\Big|\,dt\,dx \quad (1.9)$$

with

$$B(\varphi) = \Big|\det\big(id+\nabla\varphi(t)\big)\Big|\;\left(\Big[\big(\nabla(id+\varphi(t))\big)^*\Big]^{-1}\right)^*\;\Big[\big(\nabla(id+\varphi(t))\big)^*\Big]^{-1}$$

So the problem is equivalent to the partial differential equation on $\overline{y_\varphi}$:

$$\begin{cases} \partial_t\overline{y_\varphi} - \dfrac{1}{|\det(id+\nabla\varphi(t))|}\operatorname{div}(B(\varphi)\nabla\overline{y_\varphi}) = f\circ(id+\varphi) & \text{on } \Omega_0\times\mathbb{R}^+ \\ \overline{y_\varphi}(t=0)=0 & \text{on } \Omega_0 \\ \overline{y_\varphi}|_{\Gamma_0}=0 & \forall t>0 \end{cases}$$

□

THEOREM 1.3. *The problem* (1.3) *has a unique solution in* $L^2(0,T;H_0^1(\Omega_0))\cap C([0,T],L^2(\Omega_0))$

The existence and uniqueness of solutions is an application of the Lions' theorem [2, theorem X.9] which we recall in theorem 1.4

THEOREM 1.4 (Lions). *Let* $H := L^2(\Omega_0)$, $V = H_0^1(\Omega_0)$ *and* $V' = H^{-1}(\Omega_0)$ *the dual space of* V. *Let* $f \in L^2((0,T),V')$ *and* $u_0\in H$. *Consider the variational problem:*

$$\begin{cases} \langle\partial_t u, v\rangle + a(t;u(t),v) = \langle f(t),v\rangle & \text{for almost all } t\in[0,T], \forall v\in H_0^1(\Omega_0) \\ u(0) = u_0 & \text{on } \Omega_0 \end{cases} \quad (1.10)$$

Assume that $a(t;u,v)$ *satisfies the conditions:*

(i). the mapping $t\mapsto a(t;u,v)$ *is measurable*

(ii). $(u,v)\mapsto a(t;u,v)$ *is bilinear for all* $t\in[0,T]$

(iii). $|a(t;u,v)|\le M\|u\|\|v\|$

(iv). $a(t;u,v)\ge \alpha\|v\|_V^2 - C|v|_H^2$

Then there exists a unique u *such that:*

(I). $u\in L^2(0,T;H_0^1(\Omega_0))\cap C([0,T],L^2(\Omega_0))$

(II). $\partial_t u\in L^2(0,T;V')$

(III). u is solution of the variational problem (1.10).

Proof. [of theorem 1.3] We consider the energy scalar product in L^2 defined by

$$\langle f,g\rangle = \int_{\Omega_0} f(x)g(x)|\det(id+\nabla\varphi)|dx$$

In the case of equation (1.3), we have

$$a(t;u,v) = \int_{\Omega_0} B(\varphi)\nabla v\nabla\left(\frac{u}{|\det(id+\nabla\varphi)|}\right)dx.$$

which is clearly measurable in t so the condition (i) of theorem 1.4 is satisfied.

Furthermore we have $B(\varphi)$ tends to identity when $\varphi\to 0$ in $\mathcal{W}$ so for φ small enough, $|\ \|B(\varphi)\|-1\ |\leq 1/2$. We also have $|\det(id+\nabla\varphi)|\to 1$ when $\varphi\to 0$ in $\mathcal{W}$ so if we take φ small enough, $||\det(id+\nabla\varphi)|-1|\leq 1/2$; so $|\det(id+\nabla\varphi)|\geq 1/2$ and $\|B(\varphi)\|\leq 3/2$ simultaneously.

Eventually we get $\frac{u}{|\det(id+\nabla\varphi)|}$ in $H^1_0(\Omega_0)$

We have:

$$\begin{aligned}
|a(t;u,v)| &\leq \int_{\Omega_0}\left|B(\varphi)\nabla v\nabla\left(\frac{u}{|\det(id+\nabla\varphi)|}\right)\right|dx\\
&\leq 3/2\|\nabla v\|\ \left\|\nabla\left(\frac{u}{|\det(id+\nabla\varphi)|}\right)\right\|\\
&\leq 3\|v\|_{H^1_0}\ \|u\|_{H^1_0}
\end{aligned}$$

Let now prove the last inequality on a of the assumptions of Lions' theorem:

$$\begin{aligned}
a(t;v,v) &= \int_{\Omega_0} B(\varphi)\nabla v\nabla\left(\frac{v}{|\det(id+\nabla\varphi)|}\right)\\
&\geq \frac{1}{2}\int_{\Omega_0}\nabla v\nabla\left(\frac{v}{|\det(id+\nabla\varphi)|}\right)\\
&\geq \frac{1}{3}\|v\|^2_{H^1_0(\Omega_0)}
\end{aligned}$$

So eventually theorem 1.4 applies and we have the existence and uniqueness of $\overline{y_\varphi}$, so existence and uniqueness of y_φ in $L^2(0,T;H^1_0(\Omega_0))\cap C([0,T],L^2(\Omega_0))$ (resp. $L^2(0,T;H^1_0(\Omega_\varphi))\cap C([0,T],L^2(\Omega_\varphi))$. $\square$

Remark Since the solutions of these equations exist, are unique continuous and defined at $t=T$, it makes sense to get interested in $\overline{y_\varphi}(T)$ and $y_\varphi(T)$.

1.2.2. Linearized problem :. In this section we define and study the linearized problem. The linearization is performed with respect to the parameter φ, and we will only consider the Lagrangian differential, which we denote y'_φ.

Let us first recall the definition of the Lagrangian differential.

Definition 1.5. *The Lagrangian shape differential y'_φ is the unique function (when it exists) satisfying*

$$\forall\widetilde{\Omega}\Subset\Omega\ \ y_\varphi|_{\widetilde{\Omega}} = y_0|_{\widetilde{\Omega}} + y'_\varphi|_{\widetilde{\Omega}} + o(\varphi)$$

REMARK We write $A \Subset B$ if and only if $\overline{A} \subset \overset{\circ}{B}$.

PROPOSITION 1.6. *The Lagrangian shape differential y'_φ satisfies the following equation:*

$$\boxed{\begin{cases} \partial_t y'_\varphi(t,x) - \Delta y'_\varphi(t,x) = 0 & \forall (t,x) \in \mathbb{R}^+ \times \Omega_0 \\ y'_\varphi(t=0,x) = 0 & \forall x \in \Omega_0 \\ y'_\varphi = -\varphi.n \ \frac{\partial y_0}{\partial n} & \forall (t,x) \in \mathbb{R}^+ \times \partial\Omega_0 \end{cases}} \tag{1.11}$$

Proof. Let $\widetilde{\Omega} \Subset \Omega$. We have in this open set the evolution equation

$$\partial_t y_\varphi - \Delta y_\varphi = f \ \ \forall (t,x) \in \mathbb{R}^+ \times \widetilde{\Omega}$$

Since y_0 satisfies the equation:

$$\partial_t y_0 - \Delta y_0 = f \text{ on } \ \mathbb{R}^+ \times \widetilde{\Omega},$$

we necessarily have the following evolution equation for y'_φ on $\mathbb{R}^+ \times \widetilde{\Omega}$:

$$\partial_t y'_\varphi - \Delta y'_\varphi = 0 \quad \text{on} \ \ \mathbb{R}^+ \times \widetilde{\Omega} \tag{1.12}$$

Now that we have the evolution equation of the Lagrangian shape differential, let us derive the boundary conditions. First of all, it is clear that

$$y'_\varphi(t=0) = 0.$$

Let us now compute $y_\varphi|_{\partial\Omega_0}$. Formally, we consider:

$$\int_{\partial\Omega} y_\Omega(t)\psi ds$$

for any $\psi \in C^\infty(\mathbb{R}^n)$. We differentiate formally with respect to Ω, and we get the following formula (this calculation is rather classical, see for instance [1, 6.28]):

$$\begin{aligned} 0 &= \frac{\partial}{\partial\Omega}\bigg|_{\Omega=\Omega_0} \int_{\partial\Omega} y_\Omega(t)\psi ds \\ &= \int_{\partial\Omega_0} y'_\varphi(t)\psi ds + \int_{\partial\Omega_0} \varphi(t).n\frac{\partial(y_0\psi)}{\partial n} ds \\ &= \int_{\partial\Omega_0} y'_\varphi(t)\psi ds + \int_{\partial\Omega_0} \varphi(t).n(\frac{\partial y_0}{\partial n}\psi + y_0\frac{\partial\psi}{\partial n})ds \\ &= \int_{\partial\Omega_0} y'_\varphi(t)\psi ds + \int_{\partial\Omega_0} \varphi(t).n\frac{\partial y_0}{\partial n} ds \end{aligned}$$

because $y_0|_{\partial\Omega_0} \equiv 0$. Hence we have:

$$\int_{\partial\Omega_0} (y'_\varphi(t) + \varphi(t).n\frac{\partial y_0}{\partial n})\psi ds = 0 \quad \forall \psi \in C^\infty(\mathbb{R}^2)$$

So the following non homogeneous Dirichlet boundary condition follows:

$$\boxed{y'_\varphi(t)|_{\Gamma_0} = -\varphi(t).n\frac{\partial y_0}{\partial n}(t)}$$

and thus we get (1.11). □

1.2.3. Approximate controllability of the linearized problem. In this section we prove that the linearized problem is approximately controllable using Holmgren's theorem. It is clear that if $d\Lambda(0)$ was an isomorphism of $W^{2,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ on $H^1(\mathbb{R}^n)$, then we would have using the local inversion theorem a property of exact local controllability. The same property would be valid if $d\Lambda(0)$ was surjective. But the regularity introduced by the parabolic form of the equation implies that we will never have this controllability property. Indeed, the regularity of the solution is linked with the regularity of the source function f. The most interesting property we can expect is the density of $\mathcal{N} \cap \mathcal{W}$ in $\mathcal{N}$, or for the linearized operator $d\Lambda(0)$.
As a consequence, in the sequel, we try to prove a property of approximated controllability. We prove that the function $d\Lambda(0)$ actually has a dense image, which is an interesting property by itself. But nevertheless, even with this property we are not able to prove that Λ has (even locally) a dense image.
So this method fails to prove the approximated controllability. This is the reason why we address in section 2.1 the same questions for the space semi-discretization.

Holmgren's theorem [8] is based on the propagation of zeros and uses the characteristic surfaces of the underlying equation.

Lemma 1.7. *The characteristic surfaces of the heat equation are the hyperplanes $t =$ constant.*

Proof. The heat differential operator we consider is

$$P(\partial_t, \partial_{x_1}, \dots, \partial_{x_n}) = \partial_t - \sum_{k=1}^{n} \partial^2_{x_k}.$$

So the associated characteristic polynomial reads:

$$P(T, X) = T - |X|^2,$$

and its principal part is $P_2(T, X) = |X|^2 = 0$. The solution of this later equation is $\forall i = 1 \dots N$, $X_i = 0$, and its direction $(1, 0, \dots, 0)$, so eventually the characteristic surfaces are the hyperplanes $T =$ constant . □

Proposition 1.8. *Let Ω_0 an open subset of $\mathbb{R}^n$, whose boundary Γ_0 is regular (say C^1, or differentiable with bounded differential). Let γ a non-empty open subset of Γ_0.*

Then any solution of the equations:

$$\begin{cases} -\partial_t u(t,x) - \Delta u(t,x) = 0 & \forall t > 0, x \in \Omega_0 \\ u(t=0,x) = 0 & \forall x \in \Omega_0 \\ u|_{\Gamma_0}(x,t) = 0 & \forall t > 0, x \in \Gamma_0 \\ \frac{\partial u}{\partial n}\big|_\gamma = 0 & \forall t > 0, x \in \gamma \end{cases} \tag{1.13}$$

is null on Ω_0

Proof. We use Holmgren's theorem to prove this proposition. Let $P(D)$ the heat differential operator : $P(D) = \partial_t - \Delta$. Lemma 1.7 ensures us that the characteristic surfaces of $P(D)$ are the hyperplanes $t =$ constant.
Furthermore, we assumed that :

$$\begin{cases} P(D)u = 0 & \forall (t,x) \in \mathbb{R}^+ \times \Omega_0, \\ u|_{\Gamma\times\mathbb{R}^+} = 0 & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ \frac{\partial u}{\partial n} = 0 & \forall (t,x) \in \mathbb{R}^+ \times \gamma \end{cases}$$

First of all, let us write the variational formulation satisfied by the function $u \in C(0,T;H_0^1(\Omega_0))$ solution of (1.13):

$$\forall v \in H^1(\Omega_0), \int_{\Omega_0} (\partial_t u\, v + \nabla u\, \nabla v)\, dx - \int_{\Gamma_0} \frac{\partial u}{\partial n} v\, ds = 0. \tag{1.14}$$

Let now $\tilde{u}$ be a continuation of u on an open subset close to Ω_0 defined by

$$\widetilde{\Omega_0} = \Omega_0 \cup \Omega(\gamma)$$

where $\Omega(\gamma)$ is a small perturbation corresponding to a smooth modification of the boundary of Ω_0 on γ. Let $\tilde{u} \in H_0^1(\widetilde{\Omega_0})$ be the function defined by :

$$\begin{cases} \tilde{u} = u & \text{on } \Omega_0 \\ \tilde{u} = 0 & \text{on } \Omega(\gamma). \end{cases} \tag{1.15}$$

We have the following relations:

$$\begin{cases} \partial_t \tilde{u}|_{\Omega(\gamma)} & = 0 \\ \nabla \tilde{u}|_{\Omega(\gamma)} & = 0 \\ \frac{\partial \tilde{u}}{\partial n} & = 0 \text{ on } \partial\Omega(\gamma) \end{cases}$$

and hence we have :

$$\forall v \in H^1(\Omega(\gamma)), \int_{\Omega(\gamma)} \partial_t \tilde{u}\, v + \nabla \tilde{u}\, \nabla v - \int_{\partial\Omega(\gamma)} \frac{\partial u}{\partial n} v ds = 0. \tag{1.16}$$

Now, using (1.14) and (1.16), we obtain that $\tilde{u}$ satisfies the following variational problem:

$$\forall v \in H^1(\widetilde{\Omega_0}), \int_{\widetilde{\Omega_0}} \partial_t \tilde{u}\, v + \nabla \tilde{u}\, \nabla v - \int_{\partial\widetilde{\Omega_0}} \frac{\partial \tilde{u}}{\partial n} v ds = 0. \tag{1.17}$$

At this point, we are in the field of application of Holmgren's theorem, except that the subsets Ω_0 and $\Omega(\gamma)$ are not convex. Nevertheless, we can find a partition of $\widetilde{\Omega_0}$ in open discs. Those discs are convex, and the intersection between two discs is also convex so we can apply Holmgren's theorem on those discs and then the theorem is proved.

More precisely, let B be a disc from this partition. Assume that $B = B_1 \cup B_2$ and $\tilde{u}|_{B_1} \equiv 0$ (this is always possible when $\widetilde{\Omega_0}$ is connected).

- $A_1 = \mathbb{R}_+^* \times B_1$ and $\Omega_2 = \mathbb{R}_+^* \times B_2$ are two open subsets of $\mathbb{R}^{n+1}$, convexes, and such that $\Omega_2 \in \Omega_1$
- $P(D) = \partial_t - \Delta$ is a constant coefficients differential operator.
- each characteristic plane $P(D)$ intersecting Ω_2 is an hyperplane $t = \textit{constant}$ which intersects always also w_1.

- $\tilde{u}$ solution of $P(D)\tilde{u} = 0$ and satisfies also $\tilde{u}|_{\Omega_1} = 0$

So Holmgren's theorem applies and we deduce that $\tilde{u} \equiv 0$ on $\widetilde{\Omega_0}$, in particular on Ω_0 on which we have $\tilde{u} = u$, so we have indeed $u \equiv 0$ on Ω_0 $\square$

Thanks to these results, we can prove the density result previously announced for the continuous problem:

THEOREM 1.9. *Assume that there exist a non-empty subset of Γ_0 on which*

$$\forall t \in [0,T], \quad \frac{\partial y_0}{\partial n} \neq 0$$

Then $\mathcal{R} = \{y'_\varphi(T); \varphi \in \mathcal{W}\}$ is dense in $L^2(\Omega_0)$.

Proof. We prove that $\mathcal{R}^\perp = 0$. To this purpose let us consider $g \in L^2(\Omega_0)$ such that

$$\int_{\Omega_0} g\omega = 0, \, \forall \omega \in \mathcal{R}$$

and let us prove that necessarily $g = 0$. By definition, $\mathcal{R} = \{y'_\varphi(T); \varphi \in \mathcal{W}\}$ so the later condition is equivalent to :

$$\forall \varphi \in \mathcal{W} \int_{\Omega_0} g y'_\varphi(T) = 0 \tag{1.18}$$

Let us define the adjoint state $\phi \in H^1(\Omega)$ associated to g:

$$\begin{cases} -\partial_t \phi - \Delta\phi & = g \otimes \delta_{t=T} \\ \phi(t=0) & = 0 \\ \phi|_{\Gamma_0} & = 0 \end{cases}$$

with by definition

$$\forall v \in C(0,T; H^1(\Omega_0)) \langle g \otimes \delta_{t=T}, v\rangle = \int_{\Omega_0} g v(T) dx.$$

Thus we have to show that $\phi = 0$ i.e, $g = 0$. We have:

$$(1.18) \Longrightarrow \forall \varphi \in \mathcal{W} \langle -\partial_t \phi - \Delta\phi, y'_\varphi \rangle = \int_{\Omega_0} g y'_\varphi(T) = 0$$

We now integrate by parts and get

$$\begin{aligned} \int_{\mathbb{R}^+ \times \Omega_0} (-\partial_t \phi - \Delta\phi) y'_\varphi &= \int_{\mathbb{R}^+ \times \Omega_0} \phi \partial_t y'_\varphi - \int_{\Omega_0} \phi(0) y'_\varphi(0) dx \\ &+ \int_{\mathbb{R}^+ \times \Omega_0} \nabla\phi \nabla y'_\varphi - \int_{\mathbb{R}^+ \times \Gamma_0} y'_\varphi \frac{\partial \phi}{\partial n} \\ &= \int_{\mathbb{R}^+ \times \Omega_0} (\partial_t y'_\varphi - \Delta y'_\varphi)\phi + \int_{\mathbb{R}^+ \times \Gamma_0} -y'_\varphi \frac{\partial \phi}{\partial n} + \phi \frac{\partial y'_\varphi}{\partial n} \\ &= \int_{\mathbb{R}^+ \times \Gamma_0} \varphi . n \frac{\partial y_0}{\partial n} \frac{\partial \phi}{\partial n} dt ds \end{aligned}$$

We deduce that for all t in $[0,T]$, we have $\frac{\partial y_0}{\partial n}\frac{\partial \phi}{\partial n} = 0$ on Γ_0.

We assumed that on the non-negligible subset $\gamma \in \Gamma_0$ we had $\frac{\partial y_0}{\partial n} \neq 0 \; \forall t \in [0,T]$. Hence the adjoint state ϕ satisfies the following equation:

$$\begin{cases} -\partial_t \phi - \Delta\phi & = g \otimes \delta_{t=T} \\ \phi(t=0) & = 0 \\ \phi|_{\Gamma_0} & = 0 \\ \frac{\partial \phi}{\partial n}|_{\gamma\times\mathbb{R}^+} & = 0 \end{cases} \tag{1.19}$$

From Holmgren's theorem [8, theorem 5.3.3], the zeros propagate instantaneously on all the zone in contact with the characteristic surface crossing γ, so the solution ϕ is zero for all $t \in \mathbb{R}^+$. Indeed,

- for all $t < T$, the equation satisfied by ϕ is the homogeneous heat equation

  $$\partial_t\phi - \Delta\phi = 0$$

  so it is 0 from the property 1.8.
- for all $t > T$, the equation satisfied by ϕ is also the homogeneous heat equation

  $$\partial_t\phi - \Delta\phi = 0$$

  so it is null by the same argument.

Lastly, the condition

$$-\partial_t\phi - \Delta\phi = g \otimes \delta_{t=T}$$

is equivalent to say that the jump of ϕ at T,

$$[|\phi|] = \phi(T^+) - \phi(T^-)$$

is equal to g. Indeed, assume that g is piecewise continuous with a discontinuity at $t = T$. We write the variational problem associated to ϕ. Let $v \in C_0^\infty(\mathbb{R}^+, H_0^1(\Omega_0))$

$$\int_{\mathbb{R}^+\times\Omega_0} \{\partial_t\phi v + \nabla\phi\nabla v\} dx dt = \int_{\Omega_0} g v(T) dx$$

$$\int_0^T \int_{\Omega_0} \phi\partial_t v + \int_T^\infty \int_{\Omega_0} \phi\partial_t v + \int_{\mathbb{R}^+\times\Omega_0} \nabla\phi\nabla v dx dt = \int_{\Omega_0} g v(T) dx - \int_{\Omega_0} (\phi(T^+) - \phi(T^-)) v(T) dx$$

$$\int_{\mathbb{R}^+\times\Omega_0} \{\phi\partial_t v + \nabla\phi\nabla v\} dx dt = \int_{\Omega_0} [g - (\phi(T^+) - \phi(T^-))] v(T) dx$$

So g is equal at the jump of ϕ. Here, ϕ is constant, equal at 0 so necessarily $g = 0$. (QED) □

1.2.4. Conclusion. In this section we have proved that the linearized problem is approximately controllable, i.e. that it has a dense range on $H_0^1(\Omega_0)$. This result does not allow us to go any further.

Consequently, even if the density of the range of the linearized operator might be of independent interest, it does not have any relevance, a priori, for solving the

original problem in which the nonlinear mapping is involved. It is precisely for this reason that we will turn our attention to the discretized problem in section 2.1.

Nevertheless it was clear since the beginning that the exact controllability would not hold, because of the regularity of the parabolic operator. This is not the case for the hyperbolic operator, and we will show that an exact controllability property will hold.

1.3. Controllability of the wave equation. In this section we study the controllability of the wave equation with respect to the shape of the domain. We prove here that the hyperbolic problem is exactly controllable.

First of all let us define the set of admissible domains $\mathcal{U}$ for this problem. We are quite free on the space to chose for these perturbations, the only requirement is to have sufficient diversity so that the problem will be controllable. We do not look for minimal conditions, but just for sufficient conditions on the perturbations we admit.

First of all, we require that the admissible open sets should satisfy the same regularity properties as the heat equation:

$$\mathcal{W} = L^\infty(\mathbb{R}^+, \mathcal{V}) \cap C(\mathbb{R}^+, \mathcal{V}).$$

Moreover, since the "information" has a finite speed in the wave equation, the perturbations we define should be slow enough to ensure that the problem is well posed (the perturbation should stay in the characteristic "dependence cone" of slope 1). More precisely, the time differential should be in L^∞ with norm less than 1. The perturbation φ should be in $\mathcal{W}$, be time-differentiable and in the sense of distribution and its differential should be in L^∞ and $\|\partial_t \varphi\|_{L^\infty} \leq 1$.

On the other hand, at a technical level, it is easier for us to assume that the perturbation is C^1, so that the existence and uniqueness for the perturbed problem is ensured. Finally, to be able to define the space where we will invert the functions, we assume that the transformations are uniformly bounded by a given α.
Eventually, we define the following set of admissible transformations:

$$\mathcal{M} = \Big\{ \varphi \in C^1(\mathbb{R}^+, \mathcal{V}) \cap L^\infty(\mathbb{R}^+, \mathcal{V})\,;\ \|\varphi\|_\infty \leq \alpha,\ \ \partial_t \varphi \in L^\infty(\mathbb{R}^+, \mathcal{V}) \ \text{ and } \ \|\partial_t \varphi\|_\infty \leq 1 \Big\}.$$

For $\varphi \in \mathcal{M}$, we denote:

$$\Omega_\varphi(t) = (id + \varphi(t))\,(\Omega_0) = \{x + \varphi(t)(x); x \in \Omega_0\}.$$

Let $T = 2(\mathrm{diam}(\Omega_0) + \alpha)$ and $B(y, R) = \{x \in \mathbb{R}^n; |x - y| < R\}$ la ball of center y and of radius R. Let also

$$X_0^T = (\partial\Omega_0 + B(0, \alpha)).$$

$$\Omega_0^T = \Omega_0 + B(0, \alpha).$$

This is the "reachable neighborhood" of Γ_0 at time T in $\mathbb{R}^n$. When we will take a perturbation in the space we consider, all the perturbed open subsets will be inside the open set Ω_0^T. Let us finally denote $H_0'^1(\Omega_0) = \{H_0^1(\Upsilon); \partial\Upsilon \subset X_0^T\}$ and $L'^2(\Omega_0) = \{L^2(\Upsilon); \partial\Upsilon \subset X_0^T\}$

Let $f \in L^2(\mathbb{R}^+, H^{-1}(\mathbb{R}^n))$, $y^0 \in H_1^0(\Omega_0)$ and $y^1 \in L^2(\Omega_0)$ we define y_φ as the solution of the equations:

$$\begin{cases} \partial_t^2 y_\varphi(t,x) - \Delta y_\varphi(t,x) = f(t,x) & \forall t > 0, x \in \Omega_\varphi(t) \\ y_\varphi(t=0,x) = y^0(x) & \forall x \in \Omega_0 \\ \partial_t y_\varphi(t=0,x) = y^1(x) & \forall x \in \Omega_0 \\ y_\varphi(t,x) = 0 & \forall t > 0,\ x \in \Gamma_\varphi(t) \end{cases} \tag{1.20}$$

The question we address is to characterize the set of traces at $t = T$ of the solutions of this problem when φ is an admissible transformation:

$$\mathcal{R}(\mathcal{M}) = \{(y_\varphi(T), \partial_t y_\varphi(T)); \varphi \in \mathcal{M}\}.$$

Let also y_0 be the solution associated to the trivial perturbation $\varphi \equiv 0$.

Let us finally denote:

$$\Lambda : \begin{cases} \mathcal{M} \to H^1(\Omega_0^T) \times L^2(\Omega_0^T) \\ \varphi \to (y_\varphi(T), \partial_t y_\varphi(T)) \end{cases} \tag{1.21}$$

and $\mathcal{R}(\Lambda) = \{(y_\varphi(T), \partial_t y_\varphi(T))\,|\varphi \in \mathcal{M}\}$.

1.3.1. Existence and uniqueness of solutions. The same method as the one we used for the heat equation yields to the following equation:

$$\int_{\mathbb{R}^+ \times \Omega_0} \{\partial_t^2 \bar{y}_\varphi |\det(id + \nabla\varphi(t))| - \operatorname{div}(B(\varphi))\nabla \bar{y}_\varphi\} \Psi dt\, dx$$
$$= \int_{\mathbb{R}^+ \times \Omega_0} f \circ (id + \varphi)\Psi |\det(id + \nabla\varphi(t))| dt\, dx \tag{1.22}$$

where

$$B(\varphi) = \left|\det\big(id + \nabla\varphi(t)\big)\right| \left(\left[\big(\nabla(id + \varphi(t)\,)\big)^*\right]^{-1}\right)^* \left[\Big(\nabla\big(id + \varphi(t)\big)\Big)^*\right]^{-1}.$$

We can see that $B(\varphi)$ is symmetrical. It is positive because it is a perturbation of identity (when φ is small enough). We now work on $L^2(\Omega_0)$ with the dot product :

$$\langle (a,b),(c,d)\rangle_{H_0^1(\Omega_0)\times L^2(\Omega_0)} = \int_{\Omega_0} B(\varphi)\nabla a.\nabla c + \int_{\Omega_0} bd |\det(id + \nabla\varphi(t))| dx.$$

Here again we use Lions' theorem 1.4 to prove the existence and uniqueness of solution for the perturbed system.

Indeed, with Lions' notations we have:

$$a(t; u, v) = \langle u(t), v\rangle_{H_0^1(\Omega_0)}.$$

Thus a is bilinear, symmetrical, and continuous. We only have to prove that it is C^1. The time dependence is directly linked with the function φ. Taking $t \to \varphi(t)$ in C^1, we are sure that $t \to a(t)$ is C^1. So under this condition on φ we have the first assumption of Lions' theorem. The other hypothesis is the same as the one we used for the heat equation and the same proof applies.

Hence Lions' theorem 1.4 ensures us that there exists a unique $\bar{y}_\varphi$, so a unique y_φ. The solutions exist until $t = T$ and are continuous so it is possible to be interested in $\bar{y}_\varphi(T)$ and $y_\varphi(T)$.

1.3.2. Linearized problem. In this section we compute the linearized problem using the Lagrangian shape differential we defined in the heat equation section.

PROPOSITION 1.10. *The Lagrangian shape differential satisfies the following equation:*

$$\boxed{\begin{cases} \partial_t^2 y'_\varphi - \Delta y'_\varphi = 0 \\ y'_\varphi(t=0) = 0 \\ \partial_t y'_\varphi(t=0) = 0 \\ y'_\varphi|_{\partial\Omega_0} = -\varphi.n\frac{\partial y_0}{\partial n} \end{cases}} \tag{1.23}$$

1.3.3. Exact controllability of the wave equation. In this section we prove that $d\Lambda(0)$ is surjective from $\mathcal{M}$ to $H_0^{\prime 1}(\Omega_0) \times L^{\prime 2}(\Omega_0)$. The local surjectivity theorem will give us as an application the exact controllability of the problem

THEOREM 1.11. *The linearized function $d\Lambda(0)$ is surjective from $C^1([0,T],\mathcal{W}_h)$ onto $H_0^1(\Omega_0) \times L^2(\Omega_0)$*

Proof. The proof of this theorem is based on classical results on the zero controllability of the wave equation, proved by Lions in [10, 12]. In this paper, Lions considers an open bounded subset Ω of $\mathbb{R}^n$ with a smooth boundary Γ and a control time $T > 2\mathrm{diam}(\Omega)$, $\gamma \subset \Gamma$ satisfying some geometrical control conditions. The problem was to find a function v such that the solution of the equation

$$\left\{ \begin{array}{cccl} \partial_t^2 u - \Delta u & = & 0 & \text{on } \Omega \times (0,T) \\ u(0) & = & u_0 & \text{on } \Omega \\ \partial_t u(0) & = & u_1 & \text{on } \Omega \\ u(x,t) & = & v(x,t) & \text{on } \gamma \times (0,T) \end{array} \right.$$

was such that $y_v(x,T) = \partial_t y_v(x,T) = 0$.

Let us return to the proof of our theorem. We recall that the function $d\Lambda(0)$ is defined by:

$$d\Lambda(0) : \begin{cases} C^1([0,T],\mathcal{W}_h) & \longrightarrow H_0^1(\Omega_0) \times L^2(\Omega_0) \\ \psi & \longrightarrow (y'_\psi(T), \partial_t y'_\psi(T)) \end{cases}$$

where the function y'_ψ is solution of :

$$\left\{ \begin{array}{lcl} \partial_t^2 y'_\psi - \Delta y'_\psi & = & 0 \\ y'_\psi(t=0) & = & 0 \\ \partial_t y'_\psi(t=0) & = & 0 \\ y'_\psi|_{\partial\Omega_0} & = & -\psi.n\frac{\partial y_0}{\partial n} \end{array} \right.$$

Let us assume that $\frac{\partial y_0}{\partial n} \geq \varepsilon > 0$ on a subset $\gamma \subset \Gamma_0$ satisfying the geometric control condition of Lions [10]. In this case, $-\varphi.n\frac{\partial y_0}{\partial n}$ can take any value in $L^2(\gamma)$ and we can exactly control this equation for all time $T > 2(\text{ diam } (\Omega_0) + 2\alpha)$.

More precisely, there exists a certain $\psi \in \mathcal{M}$ vanishing outside γ and such that for all pair (y^0, y^1) of initial condition, we have $y_\psi(T) = y^0$ and $\partial_t y_\psi(T) = y^1$. Indeed, there exists $\tilde{\psi}$ vanishing on the boundary of Ω_0 outside γ, such that the problem

$$\begin{cases} \partial_t^2 y'_{\tilde\psi} - \Delta y'_{\tilde\psi} & = & 0 \\ y'_{\tilde\psi}(t=0) & = & y^0 \\ \partial_t y'_{\tilde\psi}(t=0) & = & y^1 \\ y'_{\tilde\psi}|_{\partial\Omega_0} & = & -\tilde\psi.n\frac{\partial y_0}{\partial n} \end{cases}$$

satisfies $y_{\tilde\psi}(T) = 0$ and $\partial_t y_{\tilde\psi}(T) = 0$. But since the wave equation is reversible, changing t into $\tau = T - t$, we obtain that $y_{\tilde\psi}(\tau) := Y(\tau)$ satisfies :

$$\begin{cases} \partial_t^2 Y - \Delta Y & = & 0 \\ Y(t=T) & = & y^0 \\ Y(t=0) & = & 0 \\ \partial_t Y(t=T) & = & y^1 \\ \partial_t Y(t=0) & = & 0 \\ Y|_{\partial\Omega_0} & = & -\tilde\psi.n\frac{\partial y_0}{\partial n} \end{cases}$$

So Y is the solution we are searching for.
□

1.3.4. Control of the wave equation. In this section we prove a theorem of exact controllability of the wave equation with respect to the shape of the domain.

THEOREM 1.12. *Let y_0 the solution of the unperturbed problem:*

$$\begin{cases} \partial_t^2 y_0 - \Delta y_0 & = & f & \text{on } \Omega_0 \\ y_0(t=0) & = & y^0 & \text{on } \Omega_0 \\ \partial_t y_0(t=0) & = & y^1 & \text{on } \Omega_0 \\ y_0|_{\Gamma_0(t)}(t) & = & 0 & \forall t > 0. \end{cases}$$

There exists a neighborhood of $(y_0(T), \partial_t y_0(T))$ in $H_0^1(\Omega_0^T) \times L^2(\Omega_0^T)$, denoted $\mathcal{N}$ such that for all $A \in \mathcal{N}$, there exists $\varphi \in \mathcal{M}$ such that $A = \Lambda(\varphi)$.

Proof. We proved that Λ was differentiable and that its differential at 0 is surjective. So the local surjectivity theorem (see e.g. Luenberger [13])proves theorem 1.12.
□

2. Control of the semi-discrete wave and heat equations. In the section 1, we addressed the problem of the controllability of the heat and wave equations with respect to the shape of the domain and proved section that the wave equation is locally exactly controllable with respect to the shape of the domain, that the parabolic problem was not exactly controllable and that its linearization was approximately controllable.

For this reason that we shall turn our attention to the discretized problem. There are two good reasons for doing this. First, it is relevant from a computational point of view and, from the mathematical point of view, since the range is finite dimensional, the density of the range of the linearized operator will imply its surjectivity and therefore allows to apply the inverse function theorem.

In this way, in the context of the discretized finite-dimensional equation, we shall prove a local exact controllability result.

2.1. Semi-discrete heat equation in a square. In this section we consider the semi-discrete heat equation on a square $[0, a] \times [0, b] \in \mathbb{R}^2$, with $a/b \in \mathbb{Q}$ (so that we can use the mesh size in the two directions). We discretize this set with a step h. The domain we consider is now

$$\Omega_h = \{m = (ih, jh); (i, j) \in \{0, \ldots, M\} \times \{0, \ldots, N\}\}.$$

Definition 2.1. *Let $m = (ih, jh) \in (\mathbb{Z}h)^2$ and the discrete neighborhood of m defined:*

$$B(m) = \{(kh, lh); (k, l) = (i, j), (i-1, j), (i+1, j), (i, j-1), (i, j+1)\}$$

The set of strict neighbors of m is $\mathcal{B}(m) = B(m) \setminus \{m\}$.

Definition 2.2. *The discrete interior of Ω_h is defined by as:*

$$\overset{\circ}{\Omega}_h = \{m \in \Omega_h; \mathcal{B}(m) \subset \Omega_h\},$$

the discrete boundary of Ω_h by

$$\Gamma_h = \Omega_h \setminus \overset{\circ}{\Omega}_h,$$

and the exterior of Ω_h as:

$$\overset{\circ}{F}_h = (\mathbb{Z}h)^2 \setminus \Omega_h.$$

Remark Those three sets form a partition of $(\mathbb{Z}h)^2$.

We assume that the only free part of the boundary is $\{(i, j); i = 0\}$. This means that the only moving part of this set is the only part of the boundary that can move to control the solution. Furthermore, the free points of the boundary will move only along the normal to this boundary.

Definition 2.3 (Function spaces). *We denote $\mathcal{F}(X)$ the set of maps $X \mapsto \mathbb{R}$ and $\mathcal{F}_0(X)$ the set of maps in $\mathcal{F}(X)$ vanishing on the boundary of X.*

We consider in this paper time dependent maps, taking values in $\mathcal{F}(\Omega_h)$. In particular we will use $C([0, T], \mathcal{F}(\Omega_h))$, the set of continuous functions $[0, T] \mapsto \mathcal{F}(\Omega_h)$ and the spaces $L^p(0, T; \mathcal{F}(\Omega_h))$.

Remark The set $\mathcal{F}(\Omega_h)$ will be identified to $\mathbb{R}^{MN}$.

2.1.1. Unperturbed state. Definition 2.4. *Let A be the finite difference operator with Dirichlet boundary conditions defined by:*

$$\begin{cases} \mathcal{F}_0(\Omega_h) & \longrightarrow \mathcal{F}(\overset{\circ}{\Omega}_h) \\ \phi & \longrightarrow A\phi \end{cases}$$

where

$$\forall m \in \overset{\circ}{\Omega}_h\ ,\ [A\phi]_m = \frac{1}{h^2}[4\phi(m) - \sum_{p \in B(m), p \neq m} \phi(p)] \tag{2.1}$$

The reference state u we consider is the solution of the equation

$$u \in \mathcal{F}_0(\Omega_h) \text{ such that } \left\{ \begin{array}{rcl} \partial_t u + Au & = & F \\ u(t=0) & = & u_0 \end{array} \right. \tag{2.2}$$

with $F \in \mathcal{F}(\overset{\circ}{\Omega}_h)$ depending on the source term f of the continuous initial problem. If f is continuous, then F will be:

$$\forall m \in \overset{\circ}{\Omega}_h \ , \ F_m = f(m).$$

If f is not continuous, for instance is $f \in L^2$ or H^1, then we can change the value of $f(m)$ by a mean value of f on a neighborhood of m.

2.1.2. The perturbed problem. As in the continuous case, we are interested in small perturbations of the shape of the domain Ω_h. Changing the shape of Ω_h consists in moving continuously the nodes of the mesh corresponding to $x = 0$. On this new subset, the finite difference Laplace operator is modified as follows:

DEFINITION 2.5. *We consider the set $\{V_j ; j = 1..N-1\}$ of vector fields $\Omega_h \mapsto \mathbb{R}^2$ by:*

$$\forall j \in \{1..N-1\}, \begin{cases} V_j(m) = (0,0) & \text{if } m \neq (0, jh) \\ V_j(m) = (1,0) & \text{if } m = (0, jh) \end{cases}$$

Let W_h be the vector space spanned by the family $(V_j)_{j\in\{1...N-1\}}$. The perturbations we consider in this problem are in the set:

$$\mathcal{W}_h = \{ \sum_{j=1}^{N-1} h\, \lambda_j(t) V_j ; t \to \lambda_j(t) \in L^\infty(\mathbb{R}^+; Wh) \cap C(\mathbb{R}^+; Wh)$$
$$\text{and such that } \sup_{j=1..N-1} \|\lambda_j\|_\infty < 1/2\} \tag{2.3}$$

REMARK Note that the perturbation has the same magnitude as h. This does not allow us even in the better cases to have the continuous case as a limit case since the perturbation tends to the trivial condition as the mesh becomes finer.

DEFINITION 2.6. *Let us define Γ^1 the first layer of interior nodes:*

$$\Gamma^1 = \{(1,j) \ ; \ j \in \{1, ..N-1\}\}.$$

Let us now define the perturbed Dirichlet operator.

DEFINITION 2.7. *Let $\varphi(t) = \sum_{j=1}^{N-1} \lambda_j(t) h V_j \in \mathcal{W}_h$.*

The operator $A(\varphi) : \mathcal{F}_0(\Omega_h^\varphi) \mapsto \mathcal{F}(\overset{\circ}{\Omega}_h)$ is defined as:

$$\begin{cases} \frac{1}{h^2}[4\phi(m) - \sum\limits_{p\in B(m),\, p\neq m} \phi(p)] & \forall m \in \overset{\circ}{\Omega}_h \setminus \Gamma^1 \\ \frac{1}{h^2}[2(1 + \frac{1}{1+\lambda_j(t)})\phi_{(1,j)} - \frac{2}{2+\lambda_j(t)}\phi_{(2,j)} - \phi_{(1,j+1)} - \phi_{(1,j-1)}] & \text{for } m = (1,j) \in \Gamma^1 \end{cases} \tag{2.4}$$

PROPOSITION 2.8. *The operator $A(\varphi)$ is bounded for all $\varphi \in \mathcal{W}_h$.*

Proof.

- For $m \in \overset{\circ}{\Omega}_h \setminus \Gamma^1$, we have $[A(\varphi)\phi]_m = [A\phi]_m = \frac{1}{h^2}[4\phi(m) - \sum_{p\in B(m),p\neq m} \phi(p)]$; so we have:

$$\begin{aligned} |[A(\varphi)\phi]_m| &\leq \frac{1}{h^2}\left[4\|\phi\|_\infty + \sum_{p\in B(m),p\neq m} \|\phi\|_\infty\right] \\ &\leq \frac{8}{h^2}\|\phi\|_\infty \end{aligned}$$

- For $m = (1,j) \in \Gamma^1$, we have

$$[A(\varphi)\phi]_m = \frac{1}{h^2}(2(1 + \frac{1}{1+\lambda_j(t)})\phi_{(1,j)} - \frac{2}{2+\lambda_j(t)}\phi_{(2,j)} - \phi_{(1,j+1)} - \phi_{(1,j-1)}),$$

and hence we have:

$$\begin{aligned} |[A(\varphi)\phi]_m| &\leq \frac{1}{h^2}\left(2(1+2)\|\phi\|_\infty + \frac{2}{2-1/2}\|\phi\|_\infty + 2\|\phi\|_\infty\right) \\ &\leq \frac{28}{3h^2}\|\phi\|_\infty \end{aligned}$$

Since we are in $\mathcal{F}(\overset{\circ}{\Omega}_h)$ which is a finite dimension vector space, all the norms are equivalent so the operator $A(\varphi)$ is bounded on $\mathcal{V}$. □

The perturbed state

Definition 2.9. *With the definitions introduced in the latest section, we define the perturbed state $\overrightarrow{u_\varphi}(x,t) \in \mathcal{F}(\overset{\circ}{\Omega}_h)$ as the unique solution in $C([0,T], \mathcal{F}(\overset{\circ}{\Omega}_h))$ of the semi-discrete problem:*

$$\begin{cases} \partial_t \overrightarrow{u_\varphi} + A(\varphi)\overrightarrow{u_\varphi} &= F \\ \overrightarrow{u_\varphi}(t=0) &= \overrightarrow{u_0} \end{cases} \tag{2.5}$$

This solution exists and is unique.

Proof. Here again we write the problem as:

$$\begin{cases} \partial_t \overrightarrow{u_\varphi} &= -A(\varphi)\overrightarrow{u_\varphi} + F \\ \overrightarrow{u_\varphi}(t=0) &= \overrightarrow{u_0} \end{cases} \tag{2.6}$$

So it is a Cauchy problem $\partial_t \overrightarrow{x} = X(\overrightarrow{x}, t)$, with $X(\overrightarrow{x}, t) = -A(\varphi(t))\overrightarrow{x} + F(t)$, in $\mathbb{R}^{(N-1)^2} \sim \mathcal{F}(\overset{\circ}{\Omega}_h)$. The function $X(\overrightarrow{x}, t)$ is measurable with respect to t; it is Lipschitz in $\overrightarrow{x}$ from proposition 2.8. So we have local existence and uniqueness of the perturbed state, which will be continuous in its definition domain. □

Proposition 2.10. *The perturbed state is defined for all $t > 0$*

Proof. Indeed, the perturbed state $\overrightarrow{u_\varphi}$ is solution of the equation

$$\partial_t \overrightarrow{u_\varphi} = F - A(\varphi)\overrightarrow{u_\varphi}$$

Let us use Gronwall's lemma to prove that the function $\overrightarrow{u_\varphi}$ cannot blow up in finite time.

$$\begin{aligned}\|\partial_t \overrightarrow{u_\varphi}\| &\leq \|A(\varphi)\overrightarrow{u_\varphi}\| + \|F\| \\ &\leq \alpha\|\overrightarrow{u_\varphi}\| + \beta\end{aligned}$$

If we assume for instance that $F \in L^\infty([0,T], W_h)$ or $F \in C([0,T], W_h)$, uniformly bounded by β on $[0,T]$. The α-constant used in the later equation is the bound of the operator A coming from the proposition 2.8.

Let ρ be the solution of the equation

$$\begin{cases}\dot{\rho} & = \alpha\rho + \beta. \\ \rho(t=0) & = \|\overrightarrow{u_\varphi}(t=0)\|\end{cases}$$

The solution of this equation is

$$\rho(t) = \Big(\rho(0) + \frac{\beta}{\alpha}\Big)e^{\alpha t} - \frac{\beta}{\alpha}$$

So Gronwall's lemma ensures us that

$$\|\overrightarrow{u_\varphi}(t)\| \leq \Big(\|u_\varphi(0)\| + \frac{\beta}{\alpha}\Big)e^{\alpha t} - \frac{\beta}{\alpha} \quad \forall t > 0$$

Eventually, the function $\overrightarrow{u_\varphi}(t)$ has a finite upper bound for all $t > 0$ so it does not blow up in finite time, so it is definite for all positive finite time. $\square$

2.1.3. Controllability of the semi-discrete heat equation. As in the continuous case, we consider the map

$$\Lambda_h : \begin{cases}\mathcal{W}_h & \longrightarrow \mathcal{F}(\overset{\circ}{\Omega}_h) \\ \varphi & \longrightarrow \overrightarrow{u_\varphi}(T)\end{cases} \tag{2.7}$$

Where $\overrightarrow{u_\varphi}(T)$ is solution of the equation (2.5). Let $Z_d = u(T)$, where u is the reference state (2.2). The problem we address is to find a neighborhood $\mathcal{V}(0) \in \mathcal{V}$ of the reference domain and $\mathcal{V}(Z_d) \in \mathcal{F}(\overset{\circ}{\Omega}_h)$ of the trace at $t = T$ of the reference solution such that $\mathcal{V}(Z_d) \subset \Lambda_h(\mathcal{V}(0))$.

In the finite-dimension spaces where the problem is now set, we prove in this section that this is possible. To this purpose, we will use the local inversion theorem. First of all we will prove that Λ_h is differentiable in the neighborhood of the origin, and that $d\Lambda_h(0)$ is surjective. Then we will use the adjoint state technique to prove that the surjectivity of $d\Lambda_h(0)$ is equivalent to a pool of conditions the semi-discrete adjoint should state satisfies. Finally, the we will prove the controllability property proving those conditions on the adjoint state, which happen to be a property of discrete unique continuation.

(i). Differentiability

Let us denote tr the trace operator:

$$\begin{cases}C([0,T], \mathcal{F}(\Omega_h)) & \mapsto \mathbb{R} \\ \phi & \mapsto \phi(T)\end{cases}$$

Then Λ_h is the composition of the trace application and the map $U : \varphi \longrightarrow \overrightarrow{u_\varphi}$.

The trace function is linear and continuous. So we only need to prove that U is Fréchet-differentiable in 0.

PROPOSITION 2.11. *$\forall \varphi \in \mathcal{W}_h$, $\forall \psi \in W_h$, the map $\varphi \mapsto \overrightarrow{u_\varphi}$ is differentiable in φ in the direction of ψ, and the Gâteaux differential $\langle D_G \Lambda_h(\varphi), \psi\rangle$, denoted $\overrightarrow{v_\varphi}(\psi)$ is solution of the differential equation:*

$$\boxed{\begin{cases} \partial_t \overrightarrow{v_\varphi}(\psi) + A(\varphi)\overrightarrow{v_\varphi}(\psi) & = -\langle \dot{A}_\varphi, \psi\rangle \overrightarrow{u_\varphi} \\ \overrightarrow{v_\varphi}(\psi)(t=0) & = \overrightarrow{0} \end{cases}} \tag{2.8}$$

Proof. First let $W_\lambda = \dfrac{u_{\varphi+\lambda\psi} - u_\varphi}{\lambda}$. The function $D_G\Lambda_h$ is the limit, when it exists, of W_λ when λ tends to 0.

(i). *Necessary condition*: We assume that this limit exists, denote it v, and we compute the equation this limit satisfies. The function $u_{\varphi+\lambda\psi}$ satisfies the equations:

$$\begin{cases} \partial_t \overrightarrow{u_{\varphi+\lambda\psi}} + A(\varphi+\lambda\psi)\overrightarrow{u_{\varphi+\lambda\psi}} & = F(t) \\ \overrightarrow{u_{\varphi+\lambda\psi}}(t=0) & = \overrightarrow{u_0} \end{cases}$$

We differentiate this equation with respect to λ at $\lambda = 0$, and we obtain the equation v,

$$\begin{cases} \partial_t v + \langle \dot{A}_\varphi, \psi\rangle \overrightarrow{u_\varphi} + A(\varphi)\overrightarrow{v} & = 0 \\ \overrightarrow{v}(t=0) & = 0 \end{cases}$$

So we deduce that if the differential of W_λ exists, then it is the solution $v_\varphi(\psi)$ of the ordinary differential equation (2.8):

$$\begin{cases} \partial_t \overrightarrow{v_\varphi}(\psi) + A(\varphi)\overrightarrow{v_\varphi}(\psi) & = -\langle \dot{A}_\varphi, \psi\rangle \overrightarrow{u_\varphi} \\ \overrightarrow{v_\varphi}(\psi)(t=0) & = \overrightarrow{0} \end{cases}$$

(ii). *Sufficient condition*: We show that the solution of the ODE (2.8) is the limit of W_λ. Indeed, the map $(\lambda_j)_{j=1\ldots N-1} \mapsto A(\phi)$ is C^∞, so the differential $\langle \dot{A}_\varphi, \psi\rangle$ is defined, and furthermore $\langle \dot{A}_\varphi, \psi\rangle \overrightarrow{u_\varphi}$ is $L^2(0,T)$. So Cauchy-Lipschitz gives us the existence and uniqueness of the solution, and we can prove that it is defined for all time using Gronwall's lemma.

Let us now show that the function $\overrightarrow{v}(t)$ defined is indeed the limit of W_λ when $\lambda \to 0$. Let $W_\lambda - v := \varepsilon_\lambda$. We have:

$$\begin{aligned} \varepsilon_\lambda &= \partial_t \varepsilon_\lambda + A(\varphi+\lambda\psi)W_\lambda - A(\varphi)v \\ &= \left[\frac{A(\varphi+\lambda\psi) - A(\varphi)}{\lambda} - \langle \dot{A}_\varphi, \psi\rangle\right] u_\varphi \end{aligned}$$

Since $A \in C^2$ in φ, the right hand of the equality is $\mathcal{O}(\lambda)$, as well as $(A(\varphi+\lambda\psi) - A(\varphi))v$ which comes from the left hand of the equality. So eventually, ε_λ satisfies the equation:

$$\boxed{\begin{cases} \partial_t \varepsilon_\lambda + A(\varphi)\varepsilon_\lambda & = o(\lambda) \\ \varepsilon_\lambda(t=0) & = \overrightarrow{0} \end{cases}}$$

So we get that

$$\|\partial_t \varepsilon_\lambda\| \le \alpha \|\varepsilon_\lambda\| + M\lambda$$

From Gronwall's lemma, we have $\varepsilon_\lambda \to 0$ when $\lambda \to 0$.
Indeed, let ρ be the solution of

$$\begin{cases} \dot{\rho} & = & \alpha\rho + \lambda M \\ \rho(t=0) & = & 0 \end{cases}$$

so we have:

$$\begin{aligned} \partial_t(\rho e^{-\alpha t}) &= (\dot{\rho} - \alpha\rho)e^{-\alpha t} \\ &= \lambda M e^{-\alpha t} \\ \rho(t)e^{-\alpha t} - \rho(0) &= -\frac{\lambda M}{\alpha}(e^{-\alpha t} - 1) \\ \rho(t) &= \lambda\Big[\frac{M}{\alpha}(e^{\alpha t} - 1)\Big]. \end{aligned}$$

And eventually, using Gronwall's lemma, we have

$$\begin{aligned} \|\varepsilon_\lambda(t)\| &\le \lambda\Big(\frac{M}{\alpha}(e^{\alpha t} - 1)\Big) \\ &\le \lambda\Big(\frac{M}{\alpha}(e^{\alpha T} - 1)\Big), \end{aligned}$$

estimation which is independent of t. So ε_λ converges uniformly[1] to 0 when $\lambda \to 0$. So the limit of W_λ exists and is indeed the function v solution of (2.8).

$\square$

Lemma 2.12. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ such that f is Gâteaux-differentiable in all directions in the neighborhood of a point x_0, with continuous differentials.*
Then f is Fréchet-differentiable at x_0 and

$$df_{x_0}.e_i = \frac{\partial f}{\partial x_i}(x_0)$$

This lemma is quite famous in differential calculus, and can be generalized in infinite dimensional cases:

Lemma 2.13 (Generalization). *Let $N : X \to Y$, and*

$$D_G N : X \to \mathcal{L}(X, Y)$$

the Gâteaux-differential of N. If $D_G N$ is definite and continuous in the neighborhood of 0, then N is Fréchet-differentiable at 0 and $D_G N = DN$, i.e :

$$\lim_{\|y\| \to 0; y \in X} \frac{\|N(x+y) - N(x) - D_G N(x)y\|}{\|y\|} = 0$$

Proposition 2.14. *The differential of u_ϕ with respect to ϕ,*

$$\varphi \longrightarrow D_G \Lambda_h(\varphi)$$

[1] We could have used the Cauchy-Lipschitz theorem with parameters to prove this last point

is continuous.

Proof. $D_G\Lambda_h(\varphi) : \psi \longrightarrow \overrightarrow{v}$ where $\overrightarrow{v}$ is solution of:

$$\left\{ \begin{array}{rcl} \partial_t \overrightarrow{v} + A_\varphi \overrightarrow{v} & = & -\langle \dot{A_\varphi}, \psi \rangle \overrightarrow{u_\varphi} \\ \overrightarrow{v}(t=0) & = & 0 \end{array} \right.$$

Hence $\overrightarrow{v}$ is solution of

$$\partial_t \overrightarrow{v} = F_{\varphi,\psi}(t, \overrightarrow{v})$$

with

$$F_{\varphi,\psi}(t, \overrightarrow{x}) = -A_\varphi \overrightarrow{x} - \langle \dot{A_\varphi}, \psi \rangle \overrightarrow{y_\varphi}.$$

We note that the function

$$\varphi \rightarrow F_{\varphi,\psi}$$

is continuous, since $\varphi \rightarrow A_\varphi$ is continuous. Cauchy-Lipschitz' theorem with parameters gives us that $\varphi \rightarrow y_\varphi$ is also continuous.

Eventually, the differential $\langle \dot{A_\varphi}, \psi \rangle$ is continuous in φ: indeed, A_φ has a rational variation in φ, and φ are such that those rational fractions have no singular point, so the dependence in φ remains continuous.

So Cauchy-Lipschitz' theorem with parameters gives us the continuity of y'_φ in φ. □

THEOREM 2.15. Λ_h *is Fréchet-differentiable at* 0.

Proof. We already proved that:

- the Gâteaux differentials in all directions of $\mathcal{V}$ exist (Prop.2.11)
- those differential are continuous (Prop. 2.14)

So lemma 2.13 gives us the Fréchet-differentiability of Λ_h at 0. □

(ii). Adjoint state technique

We recall that $\varphi \in L^2(0, T; W_h)$. We have:

$$\varphi(t) = \sum_{j=0}^{N-1} h\lambda_j(t) V_j.$$

Λ_h, defined in (2.7) is the compound of the trace function

$$tr_{t=T} : \left\{ \begin{array}{ccc} C([0,T]; \mathcal{F}(\mathring{\Omega}_h)) & \longrightarrow & \mathcal{F}(\mathring{\Omega}_h) \\ F & \longrightarrow & F(T) \end{array} \right.$$

and the map denoted U defined by:

$$U : \left\{ \begin{array}{lcl} C([0,T]; W_h) & \longrightarrow & C([0,T]; \mathcal{F}(\mathring{\Omega}_h)) \\ \varphi & \longrightarrow & u_\varphi \end{array} \right.$$

where u_φ is solution of the equations:

$$\left\{ \begin{array}{rcl} \partial_t u_\varphi + A(\varphi) u_\varphi & = & F \\ u_\varphi(t=0) & = & 0 \end{array} \right.$$

Furthermore, recall that

$$d\Lambda_h(0):\begin{cases} C([0,T],W_h) & \longrightarrow & \mathcal{F}(\overset{\circ}{\Omega}_h) \\ \varphi & \longrightarrow & y_\varphi(T) \end{cases}$$

and y_φ is solution of

$$\begin{cases} \partial_t y_\varphi + A(0)y_\varphi & = & -\langle \dot{A}(0),\varphi\rangle u_0 \\ y_\varphi(t=0) & = & 0 \end{cases}$$

Eventually, we denote $Y:\varphi\longrightarrow y_\varphi$ (so we have $d\Lambda_h(0)=tr|_{t=T}\circ Y$).

The aim of this section is to show that the differential of Λ_h at 0 is surjective.

Let $c\in\mathcal{F}(\overset{\circ}{\Omega}_h)$. It is clear that the adjoint of the trace map, tr^*c, is $c\delta_{t=T}$. We now prove that the map $d\Lambda_h(0)$ is surjective. To this purpose, we use the adjoint state technique.

$$d\Lambda_h(0) \text{ is surjective} \Leftrightarrow \{c\in\mathcal{F}(\overset{\circ}{\Omega}_h);\forall\varphi\in\mathcal{W}_h\langle d\Lambda_h(0)\varphi,c\rangle=0\}=\{0\}$$

We also have:

$$\begin{array}{ccc} & \forall\varphi\in\mathcal{W}_h & \langle d\Lambda_h(0)\varphi,c\rangle=0 \\ \Leftrightarrow & \forall\varphi\in\mathcal{W}_h & \langle tr|_{t=T}(Y(\varphi)),c\rangle=0 \end{array}$$

$$\Leftrightarrow \forall\varphi\in\mathcal{W}_h\langle Y(\varphi),tr|^*_{t=T}(c)\rangle=0 \tag{2.9}$$

Furthermore we have $Y(\varphi)=y_\varphi$ is solution of the differential equation:

$$\partial_t y_\varphi + A(0)y_\varphi = -A'_\varphi y_0$$

Definition 2.16. *The adjoint state associated to y_φ and $tr|^*_{t=T}(c)$ is the unique solution X of the equations:*

$$\begin{cases} -\partial_t X + AX & = tr|^*_{t=T}(c) \\ X(t=0) & =0 \end{cases} \tag{2.10}$$

Remark

$$\begin{aligned}\langle\partial_t y_\varphi + Ay_\varphi,T\rangle &= \langle\partial_t y_\varphi,T\rangle+\langle Ay_\varphi,T\rangle \\ &= -\langle y_\varphi,\partial_t T\rangle + \langle y_\varphi,A^*T\rangle \\ &= \langle y_\varphi,-\partial_t T+AT\rangle\end{aligned}$$

(we used that A is self-adjoint ($A^*=A$))

Using this definition we replace in (2.9) $tr|^*_{t=T}(c)$ by its expression in function of the adjoint state X defined in (2.10).

Hence we have:

$$\begin{array}{l} \forall\varphi\in\mathcal{W}_h\langle Y(\varphi),tr|^*_{t=T}(c)\rangle=0 \\ \Leftrightarrow\forall\varphi\in\mathcal{W}_h\langle Y(\varphi),-\partial_t X+AX\rangle=0 \\ \Leftrightarrow\forall\varphi\in\mathcal{W}_h\langle\partial_t y_\varphi+Ay_\varphi,X\rangle=0 \\ \Leftrightarrow\forall\varphi\in\mathcal{W}_h\langle A'_\varphi y_0,X\rangle=0 \end{array}$$

So the following theorem holds:

THEOREM 2.17. *The differential $d\Lambda_h(0)$ of Λ_h at $\varphi = 0$ is surjective if and only if we have the following unicity property: If $c \in \mathcal{F}(\overset{\circ}{\Omega}_h)$ is such that*

$$\langle X, A'_\varphi y_0 \rangle_{L^2(0,T;\mathcal{F}(\overset{\circ}{\Omega}_h))} = 0, \ \forall \varphi \in \mathcal{W}_h \tag{2.11}$$

where X is solution of :

$$\begin{cases} -\partial_t X + AX &= tr|^*_{t=T}(c) \\ X(t=0) &= 0 \end{cases}$$

Then necessarily $c = 0$.

(iii). Calculation of the differential of A at 0

PROPOSITION 2.18. *Let $j \in \{1...N-1\}$ and $\phi \in \mathcal{F}(\overset{\circ}{\Omega}_h)$. For all $\mu \in C([0,T],\mathbb{R})$, we denote $\langle A'_0, \mu(t)V_j\rangle$ the differential of A at 0 in the direction $\mu(t)V_j$. We have:*

$$[\langle A'_0, \mu(t)V_j\rangle\phi]_m = \begin{cases} 0 & \forall m \in \overset{\circ}{\Omega}_h \setminus \Gamma^1 \\ \frac{\mu(t)}{h^2}\left(\frac{1}{2}\phi_{(2,j)} - 2\phi_{(1,j)}\right) & \text{if } \ m = (1,j) \end{cases} \tag{2.12}$$

Proof. For all $j \in \{1...N-1\}$, the point $(0,j)$ of the boundary has a unique neighbor in $\overset{\circ}{\Omega}_h$, which is $(1,j)$. From the definition 2.7 of $A(\varphi)$, for all $\mu : t \to \mu(t)$ $C([0,T];\mathbb{R})$, we have :

$$\forall m \in \overset{\circ}{\Omega}_h \ \ , m \neq (1,j) \ , [A_{\mu(t)V_j}\phi]_m = [A\phi]_m$$

and we have also

$$[A(\mu(t)V_j)\phi]_{(1,j)} = \frac{1}{h^2}\left(2(1+\frac{1}{1+\mu(t)})\phi_{(1,j)} - \frac{2}{2+\mu(t)}\phi_{(2,j)} - \phi_{(1,j+1)} - \phi_{(1,j-1)}\right).$$

Hence we get

$$\begin{aligned}[A(\mu(t)V_j)\phi]_{(1,j)} - [A\phi]_{(1,j)} &= \frac{1}{h^2}[2(1+\frac{1}{1+\mu(t)} - 2)\phi_{(1,j)} - (\frac{2}{2+\mu(t)} - 1)\phi_{(2,j)}]. \\ &= \frac{1}{h^2}[2(1+1-\mu(t)+o(\mu)-2)\phi_{(1,j)} - (1-\frac{1}{2}\mu(t) - 1 + o(\mu))\phi_{(2,j)}].\end{aligned}$$

□

(iv). The condition $X|_{\Gamma^1} = 0$

DEFINITION 2.19. *The function $Y \in C([0,T],\mathcal{F}(\overset{\circ}{\Omega}_h))$ satisfies the discrete non-degeneracy condition if and only if:*

$$\forall t > 0, \forall j \in \{1,...,N-1\}, \frac{1}{2}Y_{(2,j)} - 2Y_{(1,j)} \neq 0.$$

REMARK This condition can be seen as an finite difference approximation of the condition $\frac{\partial y}{\partial n} \neq 0$. Indeed, let $y \in \mathcal{C}^2(\mathcal{V}(x))$ where $\mathcal{V}(x)$ is a neighborhood of x. Assume that y satisfies $y(x) = 0$. Then performing a Taylor expansion, we get:

$$y'(x) = \frac{1}{h}[2y(x+h) - \frac{1}{2}y(x+2h)] + o(h).$$

So the condition $\frac{1}{2}Y_{(2,j)} - 2Y_{(1,j)}$ is the finite difference expression of the continuous condition $\frac{\partial y}{\partial n} \neq 0$ we assumed in the continuous case, in theorem 1.9.
Note also that in the continuous case, we only need to assume that $\frac{\partial y}{\partial n}$ does not vanish on an open set of the boundary, and not all along the boundary. Here we need to assume the non-degeneracy condition all along the boundary of Ω_h to prove the discrete unique continuation.

Proposition 2.20. *Let us assume that the reference state satisfies the non-degeneracy condition all along the boundary of Ω_h. Then we have:*

$$X|_{\Gamma^1} \equiv 0.$$

Proof. The relation (2.11) gives us :

$$\langle X, A_0'(\mu(t)V_j)y_0\rangle_{L^2(0,T);\mathcal{F}(\overset{\circ}{\Omega}_h)} = 0,\ \forall j \in \{1,..,N-1\}, \forall \mu(t) C_0^\infty(\mathbb{R}^+,\mathbb{R}).$$

Proposition 2.18 gives us therefore that $\forall j \in \{1,..,N-1\}$,

- $\forall m \in \overset{\circ}{\Omega}_h$, $m \neq (1,j)$, $[A_0'(\mu(t)V_j)\phi]_m = 0$,
- $[A_0'(\mu(t)V_j)\phi]_{(1,j)} = \frac{1}{h^2}[\frac{1}{2}\phi_{(2,j)} - 2\phi_{(1,j)}]\mu(t)$

so we eventually have :

$$\langle X, A_0'(\mu(t)V_j)y_0\rangle_{L^2(0,T);\mathcal{F}(\overset{\circ}{\Omega}_h)} = 0,$$

Thus

$$\int_0^T \sum_{m\in\overset{\circ}{\Omega}_h} \mu(t)[A_0'(\mu(t)V_j)y_0]_m(t)X_m(t)dt = \int_0^T \mu(t)\frac{1}{h^2}[\frac{1}{2}y_0(2,j) - 2y_0(1,j)]X_{(1,j)}(t)dt = 0$$

So we deduce that for all $t > 0$, $\frac{1}{h^2}[\frac{1}{2}y_0(2,j) - 2y_0(1,j)](t)X_{(1,j)}(t) = 0$

Since $\frac{1}{2}y_0(2,j) - 2y_0(1,j)$ never vanishes, we have:

$$X_{(1,j)}(t) = 0\ \ \forall t > 0 \text{ and } \forall j \in \{1,...,N-1\}.$$

□

(v). Unique discrete continuation

The aim of this section is to prove the uniqueness condition(2.11). From proposition 2.20, and under the discrete non-degeneracy condition on the reference state y_0, we have $X|_{\Gamma^1} \equiv 0$.

We now prove this condition using the adjoint state defined by (2.16):

$$\begin{cases} -\partial_t X + AX & = & tr|^*_{t=T}(c) \\ X(t=0) & = & 0 \end{cases}$$

implies that X identically null, so $tr^*_{t=T}(c) = 0$ and $c = 0$. To this end, we study the propagation of the zeros of X on Ω_h from its boundary, as we did in the continuous case using Holmgren's theorem. The main difference is that the propagation of zeros in the continuous case is a global property, whereas it is a local property in the discrete

case. This is why we need to assume the non-degeneracy condition all along the the boundary of Ω_h.

THEOREM 2.21. *The unique solution of the equations*

$$\left\{\begin{array}{ccc} -\partial_t X + AX & = & tr|^*_{t=T}(c) \\ X(t=0) & = & 0 \\ X|_{\Gamma^0} & = & 0 \\ X|_{\Gamma^1} & = & 0 \end{array}\right. \tag{2.13}$$

is $X \equiv 0$, and so $c = 0$.

Proof.

(i). First we are interested in the equation $-\partial_t X + AX = tr|^*_{t=T}(c)$. On the sets $[0,T[$ and $]T,\infty[$, the equation simply reads $-\partial_t X + AX = 0$ so we have existence, uniqueness and continuity of the solution X in these domains. The right hand term can be interpreted as an imposed jump condition at time $t = T$. Indeed, let us write the variational formulation of the problem(2.13): Let $v \in C_0^\infty(\mathbb{R}^+, \mathcal{F}(\overset{\circ}{\Omega}_h))$. The variational formulation reads:

$$\int_0^\infty \sum_{m \in \overset{\circ}{\Omega}_h} -\partial_t X_m v_m + (AX)_m . v_m dt = \sum_{m \in \overset{\circ}{\Omega}_h} v_m(T) c_m$$

i.e :

$$\int_0^\infty \sum_{m \in \overset{\circ}{\Omega}_h} \partial_t X_m v_m + (AX)_m . v_m dt = \sum_{m \in \overset{\circ}{\Omega}_h} v_m(T) c_m$$

The imposed jump at $t = T$ reads:

$$\begin{aligned} -\partial_t X + AX &= 0 \\ X(t=0) &= 0 \\ [|X|](T) &= c \\ X|_{\Gamma^0} &= 0 \end{aligned} \tag{2.14}$$

where we denoted $[|X|](T) = X(T^+) - X(T^-)$ the jump of X at $t = T$. The variational formulation of the problem reads: $\forall v \in C_0^\infty(\mathbb{R}^+, \mathcal{F}(\overset{\circ}{\Omega}_h))$

$$\begin{aligned} 0 &= \int_0^\infty \sum_{M \in \overset{\circ}{\Omega}_h} -\partial_t X_m v_m + AX_m . v_m dt \\ &= \int_0^T \sum_{M \in \overset{\circ}{\Omega}_h} -\partial_t X_m v_m + AX_m . v_m dt + \int_T^\infty \sum_{M \in \overset{\circ}{\Omega}_h} -\partial_t X_m v_m + AX_m . v_m dt \\ &= \int_0^T \sum_{M \in \overset{\circ}{\Omega}_h} X_m \partial_t v_m + AX_m . v_m dt + \int_T^\infty \sum_{M \in \overset{\circ}{\Omega}_h} X_m \partial_t v_m + AX_m . v_m dt \\ &\quad - \sum_{M \in \overset{\circ}{\Omega}_h} (X_m(T^+) - X_m(T^-)) v_m(T) + X_m(0) v_m(0) \\ &= \int_0^\infty \sum_{M \in \overset{\circ}{\Omega}_h} X_m \partial_t v_m + AX_m . v_m dt - \sum_{M \in \overset{\circ}{\Omega}_h} [|X_m|](T) v_m(T) \end{aligned}$$

We have the same variational formulations, so the solution are identical.

(ii). Calculation of the solution X: we have $\forall j \in \{0...N\}, X_{0,j} = X_{1,j} = 0$. We reason by induction on k. Assume that on the column $k-1$ ($\{(i,j);\ i = k-1\}$) and the column k ($\{(i,j); i = k\}$) we had $X = 0$. In this case, X also vanishes on the column $k+1$.
Indeed, let $j \in \{0,..,N\}$
  (a) If $j = 0$ or $j = N$ then we have indeed $X_{(k+1,j)} = 0$ because X vanishes on Γ_h, the boundary of Ω_h.
  (b) If $j \in \{1,..,N-1\}$. Let us write the equation satisfied by $X_{(k,j)}$:

$$\begin{cases} \partial_t X_{(k,j)} + (AX)_{(k,j)} = 0 \\ \partial_t X_{(k,j)} + \frac{1}{h^2}\left[4X_{(k,j)} - X_{(k+1,j)} - X_{(k-1,j)} - X_{(k,j+1)} - X_{(k,j-1)}\right] = 0 \end{cases} \tag{2.15}$$

Since we assumed that: $X_{(k,i)}(t) = X_{(k-1,i)}(t) \equiv 0$. We reinject this condition in (2.15) and we get:

$$-X_{(k+1,j)}(t) = 0 \quad \forall t.$$

So we prove that if the solution vanishes on two consecutive columns, then the solution vanishes on all the other columns. The hypothesis being that the solution X vanishes on the column $i = 0$ and $i = 1$, we indeed proved that X is identically null.

(iii). We conclude that the solution X of this problem is time continuous, (it is constant equal to 0) and that the jump of the solution at $t = T$ is null, so $c = 0$.

□

(vi). Discrete controllability result

THEOREM 2.22. *Assume that the reference state y_0 defined by* (2.5) *satisfies the non-degeneracy discrete condition* (2.19). *Let y_φ be the solution of the perturbed state* (2.5) *and $Z_d = y_0(T)$.*

There exist neighborhoods $\mathcal{V}(0) \subset C([0,T]; W_h)$ and $\mathcal{V}(Z_d) \subset \mathcal{F}(\overset{\circ}{\Omega}_h)$ such that for all $Z \in \mathcal{V}(Z_d)$ there exists $\varphi \in \mathcal{V}(0)$ such that $y_\varphi(T) = Z$.

Proof. We prove this theorem using a local surjectivity result on the map Λ_h defined in (2.7). To this purpose, we only need to check that $d\Lambda_h(0)$ is surjective.

Form theorem 2.17, $d\Lambda_h(0)$ is surjective if:

$$\langle X, A'_\varphi y_0 \rangle_{L^2(0,T);\mathcal{F}(\overset{\circ}{\Omega}_h)} = 0, \ \ \forall \varphi \in \mathcal{W}_h$$

with

$$\begin{cases} -\partial_t X + AX & = tr|^*_{t=T}(c) \\ X(t=0) & = 0 \end{cases}$$

implies that $c = 0$. The proposition 2.20 shows that when the reference state satisfies the discrete non-degeneracy condition, then $X|_{\Gamma^1} = 0$. In theorem 2.21 we proved that this second relation implies that $X = 0$ and that $c = 0$. We deduce that $d\Lambda_h(0)$ is surjective, which achieves the proof. □

2.2. Semi-discrete wave equation in a square. In this section we proceed the same way as in the section 2.1 to prove the controllability of the waves equations. We still consider the system in two dimensions, in the set $[0,a]\times[0,b]\in\mathbb{R}^2$, discretized with a step h. The domain we consider is

$$\Omega_h = \{m = (ih, jh); (i,j) \in 0,\dots,M \times 0,\dots,N\}.$$

We already defined for $m = (ih, jh) \in (\mathbb{Z}h)^2$, $B(m)$ the *neighbors* of m, $\mathcal{B}(m)$, the discrete interior $\overset{\circ}{\Omega}_h$, the boundary Γ_h, and lastly the exterior $\overset{\circ}{F}_h$.

We assume that the only free part of the boundary is $\{(i,j); i = 0\}$.

2.2.1. Unperturbed state. We define as in the parabolic case the unperturbed state, introducing the finite differences operator A with Dirichlet boundary conditions.

$$A: \ \mathcal{F}_0(\Omega_h) \mapsto \mathcal{F}(\overset{\circ}{\Omega}_h)$$

such that

$$\forall m \in \overset{\circ}{\Omega}_h \quad [A\phi]_m = \frac{1}{h^2}\left[4\phi(m) - \sum_{p\in B(m), p\neq m} \phi(p)\right] \tag{2.16}$$

The reference state u is solution of the equation

$$\begin{cases} u & \in \mathcal{F}_0(\Omega_h) \\ \partial_t^2 u_0 + A u_0 & = F \\ u_0(t=0) & = u_0 \\ \partial_t u_0(t=0) & = u_1 \end{cases} \tag{2.17}$$

The function F is defined the same way as the in the parabolic discrete state.

2.2.2. The perturbed state. As in the continuous state, we are interested in small perturbations of the shape of the domain Ω_h. We are quite free in the choice of the admissible transformations, and only look for sufficient conditions for the exact controllability. The first assumption we make on the perturbation is that the shape of the domain will be modified only moving nodes of the mesh using C^1 transformations in time. Moreover, the only moving nodes are located on the line $x = 0$, and will move along the normal to the boundary. Because of the finite propagation speed of information in the wave equation, the problem will be well posed if the boundary does not moves faster than the information, i.e. the differential of the deformation should not have a module greater than the information propagation speed, in our case 1.

On those perturbed open sets, the operator approximating the Dirichlet Laplacian is identical to the one defined for the heat equation.

Definition 2.23. *We consider $\{V_j; j = 1..N-1\}$ the vector field $\Omega_h \mapsto \mathbb{R}^2$ defined by :* $\forall j \in \{1..N-1\}$,

- $V_j(m) = (0,0)$ *if* $m \neq (0, jh)$
- $V_j(m) = (1,0)$ *if* $m = (0, jh)$

We denote W_h the real vector space spanned by $(V_j)_{j\in\{1\dots N-1\}}$. The admissible transformations we consider lie in the space:

$$\mathcal{V}=\Big\{\sum_{j=1}^{N-1}h\lambda_j(t)V_j;\;\; t\to\lambda_j(t)\in W^{1,\infty}(\mathbb{R}^+;Wh)\cap C^1(\mathbb{R}^+;Wh)$$

$$\textit{and such that}\ \sup_{j=1..N-1}\|\lambda_j\|_\infty<1/2,\;\;\|\partial_t\lambda_j\|_\infty<1\Big\}$$

Definition 2.24. *Let $M(\varphi)$ denote the $2n\times 2n$ matrix:*

$$\begin{pmatrix} 0 & -id \\ A(\varphi) & 0 \end{pmatrix}$$

Definition 2.25. *The perturbed state, denoted $\overrightarrow{u_\varphi}(x,t)\in\mathcal{F}(\mathring{\Omega}_h)$, is the unique solution in $C([0,T],\mathcal{F}(\mathring{\Omega}_h))$ of the semi-discrete problem:*

$$\begin{cases}\partial_t^2\overrightarrow{u_\varphi}+A(\varphi)\overrightarrow{u_\varphi} & =F\\ \overrightarrow{u_\varphi}(t=0) & =\overrightarrow{u_0}\\ \partial_t\overrightarrow{u_\varphi}(t=0) & =\overrightarrow{u_1}\end{cases}\tag{2.18}$$

Proposition 2.26. *We define $U_\varphi:=\begin{pmatrix}u_\varphi\\ \partial_t u_\varphi\end{pmatrix}$. For all $\varphi\in\mathcal{V}$, $U(\varphi)$ is well defined and lies in $C([0,T],\mathcal{F}(\mathring{\Omega}_h))^2$*

Proof. [Existence et uniqueness of the solution $U_\varphi(t)$:] We can indeed write the problem as:

$$\begin{cases}\partial_t U_\varphi & =-M(\varphi)U_\varphi+F\\ U_\varphi(t=0) & =\begin{pmatrix}u_0\\ u_1\end{pmatrix}.\end{cases}$$

which is a Cauchy problem of the form $\partial_t\overrightarrow{x}=X(\overrightarrow{x},t)$, with $X(\overrightarrow{x},t)=M(\varphi(t))\overrightarrow{x}+F(t)$, in $\mathbb{R}^{(2N-2)^2}\approx\mathcal{F}(\mathring{\Omega}_h)^2$. The function $X(\overrightarrow{x},t)$ is measurable in t; it is also Lipschitz in $\overrightarrow{x}$ since A is bounded (See prop. 2.8). We can deduce the local existence and uniqueness of the solution, which in its definition domain will be continuous. ▯

Proposition 2.27. *The solutions of the later problem are defined for all $t>0$*

Proof. Gronwall's lemmas applies (See the proof of prop. 2.10). ▯

2.2.3. Controllability of the semi-discrete wave equation. Like in the continuous case, we consider the function:

$$\Lambda_h:\begin{cases}\mathcal{V} & \mapsto\mathcal{F}(\mathring{\Omega}_h)\\ \varphi & \mapsto U_\varphi(T)\end{cases}\tag{2.19}$$

where $U_\varphi(T)$ is solution of the equation (2.18). Let $Z_d=u_0(T)$, where u_0 is the reference state defined in (2.17). The question we address is the following:

Is there neighborhoods $\mathcal{V}(0)\in\mathcal{V}$ of the reference domain and $\mathcal{V}(Z_d)\in\mathcal{F}(\mathring{\Omega}_h)$ of the trace at $t=T$ of the reference state such that $\mathcal{V}(Z_d)\subset\Lambda_h(\mathcal{V}(0))$.

In this case we will also apply the local surjectivity theorem to Λ_h.

(i). Differentiability

We denote here again tr the trace function at $t = T$. The map Λ_h is the composition of the function $S : \varphi \longrightarrow U_\varphi$ and the trace function. The Fréchet-differentiability of Λ_h is equivalent to the Fréchet-differentiability of S at 0.

This differentiability is an immediate consequence of the differentiability of $A(\varphi)$. Indeed, by Cauchy-Lipschitz' theorem with parameters, if A is Fréchet-differentiable in φ, then the matrix

$$M(\varphi) = \begin{pmatrix} 0 & -id \\ A(\varphi) & 0 \end{pmatrix}$$

is differentiable in φ which gives the Fréchet-differentiability of U_φ in φ.

Proposition 2.28. *$\phi \to U_\phi$ is differentiable at 0 in the direction ψ, and the Fréchet-differential $\langle d\Lambda_h(0), \psi\rangle$, denoted Y_ψ, is solution of the differential equation:*

$$\begin{cases} \partial_t Y_\psi + M(0) Y_\psi & = & -\langle \dot{M}_0, \psi\rangle U_0 \\ Y_\psi(t=0) & = & 0 \end{cases}$$

(ii). Adjoint state technique

We have $\Lambda_h = tr_{t=T} \circ S$. Moreover, recall that

$$d\Lambda_h(0) : \begin{cases} C([0,T], W_h) & \longrightarrow & \mathcal{F}(\mathring{\Omega}_h) \\ \varphi & \longrightarrow & Y_\varphi(T) \end{cases}$$

and Y_φ is solution of

$$\begin{cases} \partial_t Y_\varphi + M(0) y_\varphi & = & -\langle \dot{M}(0), \varphi\rangle U_0 \\ Y_\varphi(t=0) & = & 0 \end{cases}$$

Eventually, we denote $L : \varphi \longrightarrow Y_\varphi$.

We have $d\Lambda_h(0) = tr|_{t=T} \circ L$.

First of all let us prove that the differential of Λ_h at 0 is surjective, using the adjoint state method.

$$d\Lambda_h(0) \text{ is surjective} \Leftrightarrow \{c \in \mathcal{F}(\mathring{\Omega}_h)^2; \forall \varphi \in \mathcal{V} \langle d\Lambda_h(0)\varphi, c\rangle = 0\} = \{0\}$$

On an other hand,

$$\begin{array}{rll} & \forall \varphi \in \mathcal{V} & \langle d\Lambda_h(0)\varphi, c\rangle = 0 \\ \Leftrightarrow & \forall \varphi \in \mathcal{V} & \langle tr|_{t=T}(S(\varphi)), c\rangle = 0 \end{array}$$

$$\Leftrightarrow \forall \varphi \in \mathcal{V} \langle S(\varphi), tr|^*_{t=T}(c)\rangle = 0 \tag{2.20}$$

Furthermore, we have $S(\varphi) = Y_\varphi$ is solution of the differential equation:

$$\partial_t Y_\varphi + M(0) y_\varphi = -M'_\varphi y_0$$

Definition 2.29. *The adjoint state* Y_φ *and* $tr|^*_{t=T}(c)$ *is the unique solution* X *of the equations:*

$$\left\{ \begin{array}{rcl} -\partial_t X + M^* X & = & tr|^*_{t=T}(c) \\ X(t=0) & = & 0 \end{array} \right. \tag{2.21}$$

With this definition, we replace (2.20) $tr|^*_{t=T}(c)$ by its expression in function of the adjoint state X (2.21).
We deduce the following theorem:

Theorem 2.30. *The differential* $d\Lambda_h(0)$ *of* Λ_h *at* $\varphi = 0$ *is surjective if and only if we have the following uniqueness property:*
If $c \in \mathcal{F}(\overset{o}{\Omega}_h)$ *is such that*

$$\langle X, M'_\varphi y_0 \rangle_{L^2(0,T;\mathcal{F}(\overset{o}{\Omega}_h))} = 0, \ \forall \varphi \in \mathcal{V} \tag{2.22}$$

with

$$\left\{ \begin{array}{rcl} -\partial_t X + M^*(0) X & = & tr|^*_{t=T}(c) \\ X(t=0) & = & 0 \end{array} \right. \tag{2.23}$$

Then necessarily $c = 0$.

(iii). Calculation of the differential of M at 0

We have:

Proposition 2.31.

$$M'(0) = \begin{pmatrix} 0 & 0 \\ A'(0) & 0 \end{pmatrix}$$

Where $A'(0)$ *is defined in 2.18.*

(iv). Condition $X|_{\Gamma^1} = 0$

Proposition 2.32. *Assume that the reference state (without the time differential) satisfies the discrete non-degeneracy condition on the whole boundary of* Ω_h*. Then the relation* (2.22) *implies that*

$$X|_{\Gamma^1} \equiv 0.$$

Proof. The time differential vanishes directly, and we are lead to the proposition 2.20. □

(v). Unique discrete continuation

The aim of this section is to show the uniqueness property (2.22).From proposition 2.32, under the discrete non-degeneracy condition on the reference state y_0, we have $X|_{\Gamma^1} \equiv 0$. We now show that this condition, together with the definition of the adjoint state (2.29):

$$\left\{ \begin{array}{rcl} -\partial_t X + M^*(0) X & = & tr|^*_{t=T}(c) \\ X(t=0) & = & 0 \end{array} \right.$$

implies that X is identically vanishing, so $tr^*_{t=T} = 0$ and $c = 0$.
To this purpose, we will study the zeros propagation of X on Ω_h from its boundaries.

Theorem 2.33. *The relations:*

$$\left\{ \begin{array}{rcl} -\partial_t X + M^*(0) X & = & tr|^*_{t=T}(c) \\ X(t=0) & = & 0 \\ X|_{\Gamma^0} & = & 0 \\ X|_{\Gamma^1} & = & 0 \end{array} \right. \tag{2.24}$$

implies that $X \equiv 0$ *and* $c = 0$.

Proof.

(i). The equation $-\partial_t X + M(0)^* X = tr|^*_{t=T}(c)$ can be interpreted the same way as the one of the heat equation.

(ii). The same method, returning to the equation with second derivatives, gives us $\partial_t^2 u_\varphi = 0$ which implies, according to the initial conditions, that $u_\varphi = \partial_t u_\varphi \equiv 0$

(iii). Hence the solution X of this problem is continuous in time (it is constant equal to 0), so the jump of the solution at $t = T$ is null, which imposes that $c = 0$.

□

(vi). Discrete controllability result

THEOREM 2.34. *Assume that the reference state* y_0 *defined in* (2.2) *satisfies the discrete non degeneracy condition 2.19. Let* Y_φ *be the solution of the perturbed equation*(2.5). *Let finally* $Z_d = Y_0(T)$. *Then there exist neighborhoods* $\mathcal{V}(0) \subset C([0,T]; W_h)$ *and* $\mathcal{V}(Z_d) \subset \mathcal{F}(\overset{\circ}{\Omega}_h)^2$ *such that* $\forall Z \in \mathcal{V}(Z_d) \exists \varphi \in \mathcal{V}(0)$ *such that* $Y_\varphi(T) = Z$.

Conclusion. In this paper we proved that the linearized heat equation was approximately controllable with respect to the shape of the domain, while the wave equation is locally exactly controllable. We addressed the same questions in the case of the semi-discrete equations in two dimensions in a square and we proved that the two types of equations are exactly controllable. Nevertheless, the methods we developed in this paper do not allow us to see the discrete control as an approximation of the continuous control in the wave equation. Another discretization method should be used to address this question, the mixed finite elements method. Indeed, we claim that one of the main obstacle to this interesting issue is the discretization method used, which does not behaves smoothly in the limit $h \to 0$. For instance we know (see [15, 9]) that in boundary control problem of the unidimensional wave equation, spurious modes with high frequency numerical oscillations appear and the observability constant tends to infinity when h tends to 0. It has been proved also that this semi-discrete model is not uniformly controllable in the limit $h \to 0$.

Nevertheless, the results of Castro and Micu in [3] are promising. They studied a system based on a mixed finite element space semi-discretization the linear 1-D wave equation with a boundary control at one extreme. They show that the controls obtained with these semi-discrete systems can be chosen uniformly bounded in $L^2(0,T)$ and in such a way that they converge to the HUM control of the continuous wave equation, i.e. the minimal L^2-norm control. This result motivates to study in contrast to the classical finite element semi-discretization a mixed finite element scheme.

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